The Generalized Hilbert Boundary Value Problem

Gueorgui S. Litvinchuk and Maurício L. Reis

Departamento de Matemática e Engenharias - Universidade da Madeira, Funchal, Portugal
and Centro de Matemática e Aplicações - Instituto Superior Técnico, Lisboa, Portugal

Abstract

In this work we study the Noether and the solvability theories of the generalized Hilbert boundary value problem.

Our main goal is to obtain the defect numbers of that problem with a direct or an inverse linear fractional Carleman shift of order 2 $(\alpha(\alpha(t)) \equiv t)$ on the unit circle $\mathbb{T}$. To this end we start by reducing the mentioned problem to a singular integral operator with shift. Afterwords we use the well known fact that this operator can be reduced to a singular integral operator (without shift) whose coefficients are matrix functions. Finally we compute the partial indices of such matrix functions (which, in this case, can be represented as a product of an Hermitean matrix function with negative determinant by diagonal rational matrix functions) and use these results to obtain the defect numbers of the initial problem.

In what concerns the Noether theory of the generalized Hilbert boundary value problem, we obtain results in the cases of a non-Carleman shift or of a Carleman shift of arbitrary order.

We also give examples of this problem, with the direct Carleman shift $\alpha(t) = -t$ and with the inverse Carleman shift $\alpha(t) = \frac{1}{t}$, for which the Gakhov-Coburn formulas are not valid.

Key Words: Boundary value problems; Singular integral operators; Shift; Factorization; Noether theory; Solvability theory.

Introduction

Let Γ be a simple closed Lyapunov curve dividing the complex plane $\mathbb{C}$ in two parts D^+ and D^- and let α be a direct or an inverse shift on Γ, such that $\alpha'(t) \neq 0,\ t \in \Gamma$ and $\alpha' \in H_\mu(\Gamma)$. The *generalized Hilbert boundary value problem* consists in finding a function

$$\Phi^+(z) = u(x,y) + iv(x,y)\ ,\ z = x + iy\ ,$$

analytic in the domain D^+, whose limit values of its real and imaginary parts belong to $H_\mu(\Gamma)$ and satisfy on Γ the condition

$$a(t)u(t) + b(t)u(\alpha(t)) + c(t)v(t) + d(t)v(\alpha(t)) = h(t)\ , \tag{1}$$

which can be written in the form

$$\mathrm{Re}\{\mathcal{A}(t)\Phi^+(t) + \mathcal{B}(t)\Phi^+(\alpha(t))\} = h(t)\ , \tag{2}$$

with $\mathcal{A}(t) = a(t) - ic(t)$ and $\mathcal{B}(t) = b(t) - id(t)$, where $a, b, c, d, h \in H_\mu(\Gamma)$ are real functions.

This problem is a generalization of the (classical) Hilbert boundary value problem (obtained if $b(t) = d(t) \equiv 0$) and was proposed by E. G. Khasabov and G. S. Litvinchuk [1]. In their papers [1] and [2] the Noetherity conditions and the index formula of problem (1) with a direct or an inverse Carleman shift of order 2 $(\alpha(\alpha(t)) \equiv t)$ on Γ were obtained. In what concerns the solvability theory, it was only established for some particular cases of degeneracy of boundary condition (1) which are reduced to the well known binomial boundary value problems of Hilbert and of Carleman type.

Preliminaries

Definition 1 *A linear bounded operator $A \in \mathcal{L}(X_1, X_2)$ is a Noetherian operator if*

(i) $\mathrm{im}A = \overline{\mathrm{im}A}$ *(i.e. A is a normally solvable).*

(ii) *The numbers $\alpha(A) = \dim\ker A$ and $\beta(A) = \dim\mathrm{coker}A$ $\left(\mathrm{coker}A = X_2/\mathrm{im}A\right)$ are finite.*

The subspaces $\ker A$ and $\mathrm{coker}A$ are called the *defect subspaces* of operator A and the numbers $\alpha(A)$ and $\beta(A)$ are called the *defect numbers* of this operator.

Definition 2 *The integer $\mathcal{I}(A) = \mathrm{ind}A = \alpha(A) - \beta(A)$ is called the index of the Noetherian operator A.*

We also notice that an equation of the form $Ax = y$ is solvable if and only if $\rho = \beta(A)$ solvability conditions are fulfilled and, in that case it has $l = \alpha(A)$ linearly independent solutions. The number $I = l - \rho$ is also called the index of the (Noether) equation $Ax = y$. We say that the numbers l and ρ satisfy the Gakhov-Coburn formulas when

$$l = \max(0, I) \quad \text{and} \quad \rho = \max(0, -I)\ . \tag{3}$$

The Noether theory of an operator consists in finding a Noetherity criterion for that operator and calculating its index.

The solvability theory of an operator includes the calculation of its defect numbers, the construction of bases for the defect subspaces, the problems of spectral theory, the determination of exact or approximate solutions for the corresponding equations and boundary value problems.

Definition 3 *Let Γ be a simple closed smooth contour dividing the closed complex plane into the domains $D^+ (\ni 0)$ and $D^- (\ni \infty)$. A factorization of a non-singular matrix function $G(t)$ relative to the contour Γ is a representation of G in the form*

$$G(t) = G^+(t)\Lambda(t)G^-(t)$$

where $G^\pm(t)$ are the boundary values of matrix functions $G^\pm(z)$, analytic and non-singular in $D^\pm$, satisfying $\det G^\pm(z) \neq 0$, respectively, $\Lambda(t) = \mathrm{diag}\{t^{\kappa_1}, t^{\kappa_2}, \ldots, t^{\kappa_n}\}$, and $\kappa_1 \geq \kappa_2 \geq \ldots \geq \kappa_n$ are integers which we call partial indices of G. Their sum $\kappa_1 + \kappa_2 + \ldots + \kappa_n = \kappa = \mathrm{Ind}_\Gamma \det G(t)$ is called the total index of G.

Let Γ be a simple oriented curve, a homeomorphism $\alpha : \Gamma \to \Gamma$ is called a *shift*. A homeomorphism α preserving (changing) the orientation on Γ is called a *direct (inverse) shift* and sometimes is denoted by α_+ (α_-).

Definition 4 *A point $\tau \in \Gamma$ is called a periodic point of the shift α with multiplicity $k \geq 1$ $(k \in \mathbb{N})$, if $\alpha_k(\tau) = \tau$ and (for $k > 1$) $\alpha_i(\tau) \neq \tau$ for $i \in \{1, 2, ..., k-1\}$ where $\alpha_i(t) = \alpha(\alpha_{i-1}(t))$ and we agree that $\alpha_0(t) \equiv t$.*

A periodic point with multiplicity one $(k = 1)$ is called a *fixed point*.

We denote $\mathcal{M}(\alpha, k)$ the set of periodic points of the shift α with multiplicity k.

Definition 5 *A shift α satisfying the condition*

$$\alpha_k(t) = t \quad \forall t \in \Gamma\ , \tag{4}$$

for some $k \geq 2$, is called a Carleman shift. A shift α satisfying the condition $\mathcal{M}(\alpha, k) \neq \Gamma$, for all $k \in \mathbb{N}$ is called a non-Carleman shift.

The least value of k for which the *Carleman condition* (4) is fulfilled is called the *order* of the shift α.

Considering the particular case $\Gamma = \mathbb{T} = \{t \in \mathbb{C} : |t| = 1\}$ we introduce the linear fractional Carleman shift of order 2 $(\alpha(\alpha(t)) \equiv t)$ of the form

$$\alpha(t) = \frac{t - \beta}{\overline{\beta}t - 1} \quad , \quad t \in \mathbb{T} \quad , \quad \beta \in \mathbb{C}\backslash\mathbb{T} \quad , \tag{5}$$

which preserves or changes the orientation depending on $|\beta| < 1$ or $|\beta| > 1$, respectively. The shift (5) admits the factorization

$$\alpha(t) = \alpha^+(t)\, t^\mu \alpha^-(t)\ , \tag{6}$$

where $\alpha^+(t) = \lambda\left(\overline{\beta}t - 1\right)^{-1}$, $\alpha^-(t) = \lambda^{-1}(t - \beta)\,t^{-1}$, $\lambda = \sqrt{1 - |\beta|^2}$, $\mu = 1$, if $|\beta| < 1$ and $\alpha^+(t) = (i\lambda)^{-1}(t - \beta)$, $\alpha^-(t) = i\lambda t\left(\overline{\beta}t - 1\right)^{-1}$, $\lambda = \sqrt{|\beta|^2 - 1}$, $\mu = -1$, if $|\beta| > 1$.

Noether theory of the generalized Hilbert boundary value problem

In [1] and [2], E. G. Khasabov and G. S. Litvinchuk used an integral representation to obtain the Noether theory of the generalized Hilbert boundary value problem (2) with a Carleman shift of order 2 $(\alpha(\alpha(t)) \equiv t)$ on Γ (see also [5]). Here we use the function ν_α defined in sections $1.3 - 1.6$ of $\S\,1$ of chapter 2 of [4] and the results exposed there to obtain the Noether theory of problem (2) when α is a non-Carleman shift or a Carleman shift of any order $k \geq 2$.

We start by considering the case of a direct shift α with a non-empty arbitrary set of periodic points with multiplicity $k \geq 1$ (i.e. for some $k \geq 1$, $\mathcal{M}(\alpha, k) \neq \emptyset$).

Theorem 6 *In the above conditions, the generalized Hilbert boundary value problem is Noetherian if*

$$\nu_\alpha(\mathcal{A}, \mathcal{B}) \neq 0 \text{ on } \Gamma\ ,$$

and, in that case, the index formula for this problem is

$$I = -\frac{2}{k}\mathrm{Ind}_\Gamma\, \nu_\alpha(\mathcal{A}, \mathcal{B}) + 1\ .$$

We note that the case of a direct Carleman shift of any order $k \geq 2$ is included in the previous theorem.

It remains to consider the case of an inverse shift. In the general case of an inverse shift α such that α_2 is a direct shift with a non-empty arbitrary set of fixed points which doesn't include all the points of the curve Γ, the Noether theory of the generalized Hilbert problem is given by the following

Theorem 7 *For a shift in the above conditions, the generalized Hilbert boundary value problem is Noetherian if*

$$\nu_{\alpha_2}\left(\mathcal{A}(t)\overline{\mathcal{A}(\alpha(t))}, \mathcal{B}(\alpha(t))\overline{\mathcal{B}(t)}\right) \neq 0 \text{ sobre } \Gamma\ ,$$

and, if this condition holds, the index of problem (2) is given by the formula

$$I = -\mathrm{Ind}_\Gamma\, \nu_{\alpha_2}\left(\mathcal{A}(t)\overline{\mathcal{A}(\alpha(t))}, \mathcal{B}(\alpha(t))\overline{\mathcal{B}(t)}\right) + 1\ .$$

Solvability theory of the generalized Hilbert boundary value problem

In this subsection we obtain the defect numbers l and ρ of the generalized Hilbert boundary value problem (2) when it is considered on the unit circle $\mathbb{T}$ and α is a direct or an inverse linear fractional Carleman shift of order 2 $(\alpha(\alpha(t)) \equiv t)$ of the form (5).

We start by introducing some identities which will be used in this section. Put

$$\begin{aligned} \Delta(t) &= \mathcal{A}(t)\mathcal{A}(\alpha(t)) - \mathcal{B}(t)\mathcal{B}(\alpha(t))\ , \\ \theta(t) &= \overline{\mathcal{A}(t)}\mathcal{A}(\alpha(t)) - \mathcal{B}(t)\overline{\mathcal{B}(\alpha(t))}\ , \\ V(t) &= \mathcal{B}(t)\overline{\mathcal{A}(\alpha(t))} - \overline{\mathcal{B}(t)}\mathcal{A}(\alpha(t))\ , \end{aligned}$$

It can be directly verified that $\mathrm{Re}V = 0$, consequently $V = iV_0$ com V_0 função real $(V_0 = -iV)$.

We observe that, using the notations introduced above, the Noetherity condition of problem (2) takes the form $\Delta(t) \neq 0$ if $\alpha = \alpha_+$ and $\theta(t) \neq 0$ if $\alpha = \alpha_-$.

The case of a direct linear fractional Carleman shift

In this subsection α is a direct linear fractional Carleman shift of order 2 $(\alpha(\alpha(t)) \equiv t)$ of the form (5) (with $|\beta| < 1$), and we introduce the weighted shift operator $(U\varphi)(t) = u(t)\varphi(\alpha(t))$, where $u(t) = -\alpha^+(t) = -\lambda\left(\overline{\beta}t - 1\right)^{-1}$ (see (6)).

We start by observing that, if $\Gamma = \mathbb{T}$, the study of the solvability theory of problem (2) can be reduced to the study of the solvability theory of the singular integral operator with shift

$$T = \left(\mathcal{A}(t)I + u^{-1}(t)\mathcal{B}(t)U\right)P_+ - \left(t\overline{\mathcal{A}(t)}I + \alpha(t)u^{-1}(t)\overline{\mathcal{B}(t)}U\right)P_-\ , \tag{7}$$

with $P_\pm = \frac{1}{2}(I \pm S)$, where S is the *operator of singular integration* and I is the identity operator.

On the other hand, as $US = SU$, the solvability theory of operator T can be obtained from the solvability theory of the singular integral operator without shift

$$M = AP_+ + BP_-\ , \quad \text{with}$$
$$A = \begin{pmatrix} \mathcal{A}(t) & u^{-1}(t)\mathcal{B}(t) \\ u(t)\mathcal{B}(\alpha(t)) & \mathcal{A}(\alpha(t)) \end{pmatrix} \text{ and } B = \begin{pmatrix} -t\overline{\mathcal{A}(t)} & -\alpha(t)u^{-1}(t)\overline{\mathcal{B}(t)} \\ -tu(t)\overline{\mathcal{B}(\alpha(t))} & -\alpha(t)\overline{\mathcal{A}(\alpha(t))} \end{pmatrix}\ .$$

Hence, it is enough to study the solvability theory of operator

$$\widetilde{M} = P_+ + CP_-\ ,$$

with

$$C = A^{-1}B = \frac{1}{\Delta(t)}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\begin{pmatrix} u^{-1}(t) & 0 \\ 0 & 1 \end{pmatrix}C_0\begin{pmatrix} 1 & 0 \\ 0 & u(t) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} t & 0 \\ 0 & \alpha(t) \end{pmatrix}\ , \tag{8}$$

where $C_0 = \begin{pmatrix} V_0(t) & i\theta(t) \\ i\overline{\theta(t)} & -V_0(\alpha(t)) \end{pmatrix}$ is an Hermitean matrix function with negative determinant. Then by virtue of Theorem 2 of [6] the partial indices of the matrix function C_0 are $m\,(\geq 0)$ and $-m$.

Now, using (8), and taking into account the factorization $\alpha(t) = \alpha^+(t)\,t\alpha^-(t)$ (see (6)) and the identity $u(t) = -\alpha^+(t)$, we obtain that the partial indices of the matrix function C are

$$\kappa_1 = \kappa + m + 1 \quad e \quad \kappa_2 = \kappa - m + 1\ ,$$

where

$$\kappa = \frac{1}{2\pi}\left\{\arg\overline{\Delta(t)}\right\}_\Gamma = \mathrm{Ind}_\Gamma\overline{\Delta(t)}\ .$$

Finally, using Theorem 2.10 of [3], we obtain the following result about the solvability theory of the generalized Hilbert boundary value problem (2) with a direct linear fractional Carleman shift on $\mathbb{T}$.

Theorem 8 *In the above conditions, the numbers l, of linearly independent solutions, and ρ, of solvability conditions of the generalized Hilbert boundary value problem (2) with a direct linear fractional Carleman shift of order 2 on $\mathbb{T}$ are given by:*

1. $l = \max(0, \kappa + 1)$ *and* $\rho = \max(0, -\kappa - 1)$, *if* $\kappa_1 \leq 0$ *or* $\kappa_2 > 0$,
2. $l = \dim\ker T = \frac{\kappa_1}{2} = \frac{\kappa + m + 1}{2}$ *and* $\rho = \frac{-\kappa + m - 1}{2}$, *if* $\kappa_1 > 0,\ \kappa_2 \leq 0$ *and* κ_1 *is even,*
3. $l = \left[\frac{\kappa_1}{2} + \frac{1 - \varepsilon}{4}\right] = \begin{cases} \frac{\kappa + m}{2} & , \text{se } \varepsilon = 1 \\ \frac{\kappa + m}{2} + 1 & , \text{se } \varepsilon = -1 \end{cases}$ *and* $\rho = \begin{cases} \frac{-\kappa + m}{2} - 1 & , \text{se } \varepsilon = 1 \\ \frac{-\kappa + m}{2} & , \text{se } \varepsilon = -1 \end{cases}$, *if* $\kappa_1 > 0,\ \kappa_2 \leq 0$ *and* κ_1 *is odd.*

Now we present an example (see also [2], and [5], example 23.1) of a generalized Hilbert boundary value problem for which the Gakhov-Coburn formulas (3) are not always valid.

Example 9 Let us consider the generalized Hilbert problem

$$\mathrm{Re}\left\{\cos(k\varphi)\Phi^+(t) - i\sin(k\varphi)\Phi^+(-t)\right\} = h(t) \quad , \quad t = e^{i\varphi} \in \mathbb{T}\ , \text{with } k \in \mathbb{Z}\ . \tag{9}$$

In this case (see (2))

$$\begin{aligned} \alpha(t) &= -t\ ,\ \mathcal{A}(t) = \cos(k\varphi) = \mathrm{Re}t^{-k}\ ,\ \mathcal{B}(t) = -i\sin(k\varphi) = i\mathrm{Im}t^{-k} \\ \Delta(t) &= (-1)^k\ ,\ \theta(t) = (-1)^k\frac{t^{2k} + t^{-2k}}{2}\ ,\ V(t) = -(-1)^k\frac{t^{2k} - t^{-2k}}{2}\ . \end{aligned}$$

From this we conclude that problem (9) is Noetherian $(\Delta(t) \neq 0)$ and, as $\kappa = \mathrm{Ind}_\mathbb{T}\overline{\Delta(t)} = 0$, its index is given by $I = \kappa + 1 = 1$. Besides, for this example, according to (8) and using the factorization of a circulant matrix exposed in [7], p. 159, the matrix function C comes

$$C = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} t^{2k+1} & 0 \\ 0 & t^{-2k+1} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\ .$$

Hence, the partial indices of the matrix function C are given by

$$\kappa_1 = 2|k| + 1 \quad \text{e} \quad \kappa_2 = -2|k| + 1\ ,$$

Finally, using Proposition 2.2 of [3] to compute the number ε, it comes that, if $k \geq 0$, $\varepsilon = -1$ and if $k < 0$, $\varepsilon = 1$. Thus, by virtue of Theorem 8, we obtain

$$\begin{cases} l = k + 1 \quad \text{and} \quad \rho = k & , \text{if } k \geq 0 \\ l = -k \quad \text{and} \quad \rho = -k - 1 & , \text{if } k < 0 \end{cases}\ .$$

The case of an inverse linear fractional Carleman shift

In what follows α is an inverse linear fractional Carleman shift α of order 2 $(\alpha(\alpha(t)) \equiv t)$ of the form (5) (with $|\beta| > 1$).

Now we consider the operator U defined by $(U\varphi)(t) = u(t)\varphi(\alpha(t))$, with $u(t) = \alpha^-(t)\,t^{-1} = \frac{i\lambda}{\overline{\beta}t - 1}$ (see (6)).

As we have seen in the previous subsection, the study of the solvability theory of problem (2) can be reduced to the study of the solvability theory of operator (7).

Analogously to what happened in the case of a direct shift, now, as $US = -SU$, the solvability theory of operator (7) can be obtained from the solvability theory of the singular integral operator without shift

$$\widetilde{M} = P_+ + CP_-\ ,$$

with

$$C = A^{-1}B = \frac{1}{\alpha(t)\overline{\theta(t)}}\begin{pmatrix} i\alpha(t)u^{-1}(t) & 0 \\ 0 & -i \end{pmatrix}C_0\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} tu(t) & 0 \\ 0 & 1 \end{pmatrix}\ , \tag{10}$$

where, and $C_0 = \begin{pmatrix} V_0(t) & i\overline{\Delta(t)} \\ i\Delta(t) & V_0(\alpha(t)) \end{pmatrix}$ is an Hermitean matrix function with negative determinant.

Now, by virtue of Theorem 2 of [6] we have that the partial indices of the matrix function C_0 are $m\,(\geq 0)$ and $-m$.

Using the factorization $\alpha(t) = \alpha^+(t)\,t^{-1}\alpha^-(t)$ and the fact that $u(t) = \alpha^-(t)\,t^{-1}$, from (10) we conclude that the partial indices of the matrix function C are

$$\kappa_1 = \kappa + 1 + m \quad \text{and} \quad \kappa_2 = \kappa + 1 - m\ .$$

where

$$\kappa = \frac{1}{2\pi}\{\arg\theta(t)\}_\Gamma = \mathrm{Ind}_\mathbb{T}\theta(t)\ ,$$

Finally, by virtue of Theorem 1 of section 21 of [5] (p. 239), we obtain the following result on the defect numbers of the generalized Hilbert boundary value problem (2).

Theorem 10 *In the above conditions, putting $I = \mathrm{ind}T = \kappa + 1$, the numbers l, of linearly independent solutions, and ρ, of solvability conditions of the generalized Hilbert boundary value problem (2) with an inverse linear fractional Carleman shift of order 2 of the form (5) on $\mathbb{T}$ are given by:*

1. $l = \max(0, I)$ *and* $\rho = \max(0, -I)$, *if* $\kappa_1 \leq 0$ *or* $\kappa_2 > 0$,
2. $l = \frac{\kappa_1}{2} = \frac{\kappa + 1 + m}{2}$ *and* $\rho = \frac{-\kappa - 1 + m}{2}$, *if* $\kappa_1 > 0$ *and* $\kappa_2 \leq 0$ *and* κ_1 *is even,*
3. $l = \frac{\kappa_1 - \varepsilon}{2} = \frac{\kappa + 1 + m - \varepsilon}{2}$ *and* $\rho = \frac{-\kappa - 1 + m - \varepsilon}{2}$, *if* $\kappa_1 > 0$ *and* $\kappa_2 \leq 0$ *and* κ_1 *is odd, where* $\varepsilon = \pm 1$.

The previous theorem is also valid if we consider the shift $\alpha(t) = \frac{1}{t}$. First of all we observe that this is an inverse Carleman shift $(\alpha(\alpha(t)) \equiv t)$ on $\mathbb{T}$, but it is not a particular case of a linear fractional Carleman shift of the form (5). Afterwords, if together with this shift we consider the so called *flip operator* U, defined by $(U\varphi)(t) = \frac{1}{t}\varphi\left(\frac{1}{t}\right)$ it is easy to show that all the results obtained above for the case of an inverse linear fractional Carleman shift of the form (5) remain valid in these conditions. In fact, it is enough to replace, in the above procedure, $u(t)$ for $u(t) = t^{-1}$ and to observe that in this case, as $\alpha(t) = t^{-1}$, we have that $\alpha^+(t) = \alpha^-(t) = 1$ (see (6)).

Finally we present an example (see also [2] and [5] (example 23.2)) of the generalized Hilbert boundary value problem with the shift $\alpha(t) = \frac{1}{t}$, for which, in general, the defect numbers l and ρ do not fulfill the Gakhov-Coburn formulas (3).

Example 11 We consider the generalized Hilbert boundary value problem

$$\mathrm{Re}\left\{t^{-k}\left[u(t) + iv\left(\frac{1}{t}\right)\right]\right\} = h(t) \quad , \quad t = e^{i\varphi} \in \mathbb{T}\ , \text{with } k \in \mathbb{Z}\ . \tag{11}$$

Now (see (2))

$$\begin{aligned} \alpha(t) &= \frac{1}{t}\ ,\ \mathcal{A}(t) = \cos(k\varphi) = \mathrm{Re}t^{-k}\ ,\ \mathcal{B}(t) = -i\sin(k\varphi) = i\mathrm{Im}t^{-k} \\ \theta(t) &= 1\ ,\ \Delta(t) = \frac{t^{2k} + t^{-2k}}{2}\ ,\ V(t) = -\frac{t^{2k} - t^{-2k}}{2}\ . \end{aligned}$$

In particular we observe that, as $\theta(t) = 1 \neq 0$, problem (11) is Noetherian. Besides, $\kappa = \mathrm{Ind}_\mathbb{T}\theta(t) = 0$ so the index of this problem is $I = \kappa + 1 = 1$.

In this case, from (10) and using the factorization of a circulant matrix (see e.g. [7], p. 159), we obtain

$$C = \frac{1}{2}\begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} t^{2k+1} & 0 \\ 0 & t^{-2k+1} \end{pmatrix}\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}\ .$$

Hence we conclude that the partial indices of the matrix function C are:

$$\kappa_1 = 2|k| + 1 \quad \text{e} \quad \kappa_2 = -2|k| + 1\ ,$$

It remains to obtain the number ε. Proceeding as in example 9 we obtain that, if $k \geq 0$, $\varepsilon = -1$ and if $k < 0$, $\varepsilon = 1$. Thus, applying Theorem 10, we conclude that

(i) $l = \kappa + 1$ and $\rho = \kappa$, if $k \geq 0$,

(ii) $l = -k$ and $\rho = -k - 1$, if $k < 0$.

References

[1] Khasabov, E. G., Litvinchuk, G. S., *On Hilbert boundary value problem with a shift.* Dokl. Akad. Nauk SSSR, 142(6), 274-277, 1962 (in Russian).

[2] Khasabov, E. G., Litvinchuk, G. S., *On the index of generalized Hilbert boundary value problem.* Uspekhi matem. nauk, 20(2), 124-130, 1965 (in Russian).

[3] Kravchenko, V. G., Lebre, A. B., Rodríguez, J. S., *Factorization of singular integral operators with a Carleman shift and spectral problems.* Journal of Integral Equations and Applications, 13(4), 339-383, Winter 2001.

[4] Kravchenko, V. G., Litvinchuk, G. S., *Introduction to the theory of singular integral operators with shift.* Kluwer Academic Publishers: Dordrecht-Boston--London, 1-288, 1994.

[5] Litvinchuk, G. S., *Solvability Theory of Boundary Value Problems and Singular Integral Equations with Shift.* Kluwer Academic Publishers: Dordrecht-Boston--London, 1-378, 2000.

[6] Litvinchuk, G. S., Spitkovsky, I. M., *Sharp estimates of the defect numbers of a generalized Riemann boundary value problem.* Dokl. Akad. Nauk SSSR, 255(5), 1042-1046, 1980 (in Russian). English transl.: Soviet Math. Dokl. 22(3), 781-785, 1980.

[7] Litvinchuk, G. S., Spitkovsky, I. M., *Factorization of measurable matrix functions.* Academie-Verlag: Berlin and Birkhäuser Verlag: Basel-Boston-Stuttgart, 1-372, 1987.